1. Consider the simplicial complex \( \Delta = \{\emptyset, a, b, c, d, e, ab, ac, bc, bd, cd, abc\} \) over the set \( X = \{a, b, c, d, e\} \). The 5-dimensional Boolean lattice \( 2^X \) is shown below.

   (i) Sketch the simplicial complex and find the maximal faces.
   (ii) Circle each node in the Boolean lattice \( 2^X \) corresponding to a face of \( \Delta \), and additionally shade in those faces that are maximal.
   (iii) Box each node that corresponds to a non-face, and shade in those that are minimal.
   (iv) Find the Stanley-Reisner ideal \( I_{\Delta^c} \) and compute its primary decomposition.

\[
\begin{align*}
\text{abcde} & \quad \text{abcd} \\
\text{abce} & \quad \text{abde} \\
\text{acde} & \quad \text{bcde} \\
\text{abc} & \quad \text{abd} \\
\text{abe} & \quad \text{acd} \\
\text{ace} & \quad \text{ade} \\
\text{bcd} & \quad \text{bce} \\
\text{bde} & \quad \text{cde} \\
\text{ab} & \quad \text{ac} \\
\text{ad} & \quad \text{ae} \\
\text{bc} & \quad \text{bd} \\
\text{be} & \quad \text{cd} \\
\text{de} & \quad \text{e} \\
\phi & \quad \text{∅}
\end{align*}
\]

2. Recall from lecture that we reverse-engineered the model space of the following time-series over \( \mathbb{F}_2 \):

\[
(0, 0, 1) \xrightarrow{f} (1, 0, 1) \xrightarrow{f} (1, 1, 1) \xrightarrow{f} (1, 1, 0) \xrightarrow{f} (0, 1, 0) \xrightarrow{f} (0, 0, 0).
\]

In this problem, you will reverse-engineer the wiring diagram.

(a) Do the following steps for each \( k = 1, 2, 3 \).
   i. Write down the corresponding set of data
      \[
      \mathcal{D}_k := \{(s_1, t_{1k}), \ldots, (s_5, t_{5k})\}
      \]
      that arises from the \( k\)th coordinate of this time-series.
   ii. Compute the monomials \( m(s_i, s_j) \) for which \( t_{ik} \neq t_{jk} \), and find the ideal \( I_{\Delta^c_k} \) of non-disposible sets.
   iii. Use a computer program to find the primary decomposition of \( I_{\Delta^c_k} \).
   iv. Find all min-sets of \( \mathcal{D}_k \), and sketch a wiring diagram for each. Only include edges incident to \( x_k \).

(b) Repeat Steps (ii)–(iv) from Part (a) but to find the signed min-sets. That is, use the pseudomonomials \( p(s_i, s_j) \) to compute the ideal \( J_{\Delta^c_k} \) of signed non-disposible sets.
3. Consider the following time series of a 3-node local model over $\mathbb{F}_3$:

$$(1, 1, 1) \xrightarrow{f} (2, 0, 1) \xrightarrow{f} (2, 0, 0) \xrightarrow{f} (0, 2, 0) \xrightarrow{f} (0, 2, 2).$$

For reference, here are the input vectors $s_i$ and output vectors $t_i$:

\[
\begin{align*}
    s_1 &= (s_{11}, s_{12}, s_{13}) = (1, 1, 1), & t_1 &= (t_{11}, t_{12}, t_{13}) = (2, 0, 1), \\
    s_2 &= (s_{21}, s_{22}, s_{23}) = (2, 0, 1), & t_2 &= (t_{21}, t_{22}, t_{23}) = (2, 0, 0), \\
    s_3 &= (s_{31}, s_{32}, s_{33}) = (2, 0, 0), & t_3 &= (t_{31}, t_{32}, t_{33}) = (0, 2, 2), \\
    s_4 &= (s_{41}, s_{42}, s_{43}) = (0, 2, 2), & t_4 &= (t_{41}, t_{42}, t_{43}) = (0, 2, 2).
\end{align*}
\]

(a) Find polynomials $f_1, f_2, f_3$ in $\mathbb{F}_3[x_1, x_2, x_3]/\langle x_1^3 - x_1, x_2^3 - x_2, x_3^3 - x_3 \rangle$ that fit the data. That is, $f_j(s_i) = t_{ij}$ for all $i = 1, 2, 3, 4$.

(b) For each $i = 1, 2, 3, 4$, write down the ideal $I_i = I(s_i)$ of polynomials that vanish on the data point $s_i$.

(c) Compute a Gröbner basis $G$ for the ideal $I$ of polynomials that vanish on all of the input data points. You may use Sage, Singular, or Macaulay2, but use the graded reverse lexicographical monomial order $\text{GRevLex}$ (this is the default).

(d) Write the model space of the time series using your answer to Part (a) as the particular solution.

(e) Compute the normal form of $f_1, f_2, f_3$ with respect to $G$ by reducing them modulo the ideal $I$. Write the model space using this particular solution.

(f) Compare the sizes of the model space and the vanishing ideal.