
Exercises.

1. In this exercise, you will construct the finite field of order 9.
   
   (a) Find an irreducible polynomial of degree 2 in $\mathbb{F}_3[x]$. Note that any such $f \in \mathbb{F}_3[x]$ for which $f(c) \neq 0$ for all $c \in \mathbb{F}_3 = \{0, 1, 2\}$ will work.

   (b) Write down all 9 elements of $\mathbb{F}_9 \cong \mathbb{F}_3[x]/I$, where $I = \langle f \rangle$ is the ideal generated by the polynomial you found in Part (a). All of the elements should be of the form $g + I$, for some $g \in \mathbb{F}_3[x]$.

   (c) Construct the addition table of $\mathbb{F}_9$ and the multiplication table of $\mathbb{F}_9^* := \mathbb{F}_9 \setminus \{0\}$, like what we did for $\mathbb{F}_4$ in class. You should omit the “$+ I$” for clarity of notation.

2. Consider the reactions where two substrates $S$ and $T$ compete for binding to an enzyme $E$ to produce two different products $P$ and $Q$:

   $$
   E + S \xrightarrow{p_1} ES \xrightarrow{p_3} P + E,
   $$

   $$
   E + T \xrightarrow{q_1} ET \xrightarrow{q_3} Q + E.
   $$

   Assume that each reaction follows the Michaelis-Menten kinetics. Also, assume that that the initial enzyme concentration is $E_0 = [E] + [ES] + [ET]$.

   (a) Derive rate equations for $P$ and $Q$ in this system in terms of $[ES]$ and $[ET]$. That is, determine $d[P]/dt$ and $d[Q]/dt$.

   (b) Derive rate equations for $ES$ and $ET$.

   (c) Assume that the enzyme-substrate complexes reach equilibrium quickly: $d[ES]/dt \approx 0$ and $d[ET]/dt \approx 0$. Solve for $[E]$ in each of these equations.

   (d) Equate the two expressions for $[E]$ from Part (c) and solve for $[ET]$.

   (e) Solve for $[ES]$ by plugging your answers to Parts (c) and (d) into $E_0 = [E] + [ES] + [ET]$. You should not have $[E]$ or $[ET]$ in your final answer.

   (f) Plug this into the original ODE for $d[P]/dt$.

   (g) Repeat the previous three steps but solve for $[ES]$ instead of $[ET]$.

3. Recall our original 3-variable Boolean model of the *lac* operon:

   $$
   f_M = \overline{G_e} \land (L \lor L_e),
   $$

   $$
   f_E = M,
   $$

   $$
   f_L = \overline{G_e} \land ((E \land L_e) \lor (L \land \overline{E})).
   $$
For each of the 4 possible initial conditions, \( G_e, L_e \in \mathbb{F}_2^2 \), the model had one connected component with the biologically correct fixed point. Compute the probability that this would have happened purely by chance. (Assume a uniform distribution.)

4. Recall another model of the \( lac \) operon:

\[
\begin{align*}
    f_M &= A, \\
    f_B &= M, \\
    f_A &= (B \land L_m) \lor L \lor (A \land \overline{B}).
\end{align*}
\]

Here, \( L \) and \( L_m \) are parameters corresponding to

- High lactose: \( L = 1, L_m = 1 \).
- Medium lactose: \( L = 0, L_m = 1 \).
- Low lactose levels: \( L = 0, L_m = 0 \).

We’ll ignore any state for which \( L = 1, L_m = 0 \).

Does this model exhibit bistability? Why or why not?